

1/f-noise Generated by Constrained Diffusion through a Flux-controlling Boundary

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Calculations for constrained (one-dimensional) diffusion of current carriers in an external field are made, which concern the spectral density of fluctuations resulting from the relaxation of spatial concentration perturbations. Two different model calculations are presented: *First* diffusion in an infinite medium. In this case the spectral density of *concentration* fluctuations shows a $1/f$ behaviour, but the spectral density of *current* fluctuations does not. *Second* the diffusion in a semi-infinite medium is treated. The flux through the boundary (e.g. pore end) is assumed to be proportional to the concentration of current carriers at this boundary. In this case concentration as well as current exhibit $1/f$ behaviour. The results indicate that the existence of flux controlling boundaries (contacts) may be decisive for the occurrence of $1/f$ noise.

1. Introduction

$1/f$ noise ('excess noise', 'Flicker noise') has been observed in a great number of different physical systems (for references see e.g. ^{1–3}). The power spectrum $S(f)$ of some fluctuating physical variable $X(t)$ is called " $1/f$ -like", if $S(f)$ is varying $\propto 1/f^2$ ($\alpha \approx 1$) over a range of at least some orders of magnitude in frequency f . Up to now no generally accepted theory of excess noise exists. Moreover it is not clear, if the occurrence of excess noise in so many different physical devices may be explained by one general theory. In this paper we present model calculations based on classical diffusion processes, which might indicate a possibility to find a general theory for at least a great number of different excess noise sources such as semiconductor diodes, transistors, carbon microphones, thin films, or biological and artificial membranes.

A possible approach to a theoretical understanding of $1/f$ noise consists in the investigation of the time dependence of elementary noise events which leads to the desired $1/f$ frequency dependence of spectral intensity $S(f)$. Schönfeld ⁴ has shown that a $1/\sqrt{t}$ dependence yields $1/f$ noise. On the other hand it is known ^{5,6} that by a superposition of mutually independent usual relaxation processes $e^{-t/\tau}$ with different time constants τ a $1/f$ spectrum can be obtained, if the time constants have a distribution proportional to $1/\tau$.

The latter result has been used by McWorther ⁷ in his theory of $1/f$ noise in metal-oxide-silicon transistors based on an electron trapping model. Handel ⁸

has given a $1/f$ noise theory based on turbulences of the current carriers, where also the obtained $1/f$ spectrum is a continuous superposition of contributions with different time constants arising from eddies of different size. Lundström and McQueen ⁹ have proposed a mechanism for the observed $1/f$ noise in nerve cell membranes ^{10,11}, assuming that the conductance of the membrane pores is modulated by vibrations of the hydrocarbon chains of the lipid molecules constituting the membrane. Recently Neumcke ¹² has proposed a theory of $1/f$ membrane noise generated by diffusion polarization processes in the unstirred solution layers near the membrane. In this theory the author uses the fact that fluctuations in membrane permeability exhibit a $1/\sqrt{t}$ time course of relaxation current due to diffusion polarization ¹³.

The main stimulation for performing the calculations presented in our paper has come from recent experiments by Dorset and Fishman ¹⁴, who have observed $1/f$ current noise under applied $d-c$ current in micropipettes, single pore membranes, multipore and polymer mesh membranes. Because of these experimental results we have decided to reexamine the old idea that the $1/f$ characteristic of excess noise is generated by diffusion-like processes (see e.g. ^{15–20}). We make the following basic assumptions:

1. The $1/f$ characteristic of the spectral density is generated by diffusion-controlled relaxation processes of the current carriers themselves.
2. The sources of the fluctuations are spatial density fluctuations of the current carriers.
3. The diffusion is *one-dimensional* diffusion in an *external* field.

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4. The quantity of interest is the flux of current carriers through the *boundary layer* between two different media (e.g. pore and aqueous solution).

In contrast to older theories¹⁵⁻²⁰ we take into account the influence of an external field and of the boundary ('contact') on the diffusion process. Indeed our calculations will show that the latter point may be essential for the occurrence of 1/f type noise. The assumption that dimension might play an important role is suggested by the well known fact that the relaxation of a (delta-shaped) concentration fluctuation in an infinite medium is just 1/Vt-like in the one-dimensional case and 1/Vt² (1/Vt³)-like in the two (three) dimensional cases (see e.g.^{2, p. 54}).

We shall present two different model calculations: *First*, the relaxation of spatial concentration fluctuations in an infinite medium (a pore). In this case the resulting spectral density of concentration fluctuations shows 1/f behaviour, but the spectral density of resulting current fluctuations does not. In the *second* model calculation we treat the case of one-dimensional diffusion in a semi-infinite medium under the assumption that the flux through the boundary (e.g. pore end) is proportional to the concentration of current carriers at this boundary. In this case the spectral density of current and concentration fluctuations is 1/f-like. These results indicate that indeed according to assumption 4 the existence of a *flux-controlling boundary* between two media may be decisive for the occurrence of 1/f noise.

2. Theoretical Preliminaries

a) Carson's Theorem

As is well known the spectral density $S(f)$ of a stationary random process $X(t)$ is related to the autocorrelation function $\overline{X(t)X(t+s)}$ by the Wiener-Khinchine theorem^{1, 21, 22}

$$S(f) = 4 \int_0^{\infty} \overline{X(t)X(t+s)} \cos \omega s \, ds \quad (1)$$

($\omega = 2\pi f$). If $X(t)$ is the sum of a large number of independent events $F_i(t)$ occurring at random at the average rate λ , it is usually easier to calculate $S(f)$ directly from the time course of $F_i(t)$ with the use of Carson's theorem (see e.g.^{1 or 4}) which can be stated as follows: Let $X(t)$ be

$$X(t) = \sum_i F_i(t), \quad F_i(t) = F(t - t_i) \quad (2)$$

where $F_i(t) = 0$ for $t < t_i$ and $F(t - t_i)$ for $t \geq t_i$ represents the effect of an event starting at t_i . Then with the Fourier transform $\psi_F(f)$ of $F(u)$

$$\psi_F(f) = \int_{-\infty}^{+\infty} F(u) e^{-j\omega u} \, du \quad (3)$$

the spectral density of $X(t)$ is

$$S(f) = 2\lambda |\psi_F(f)|^2 = 2\lambda (\text{Re } \psi_F^2 + \text{Im } \psi_F^2). \quad (4)$$

If $X(t)$ is composed by a sum of *independent* events, which are not identical, (4) may be extended to:

$$S(f) = 2\lambda \overline{|\psi_F(f)|^2} \quad (4a)$$

where $\overline{|\psi_F(f)|^2}$ is the average value of $|\psi_F(f)|^2$. We intend to apply Carson's theorem for a model noise system where the elementary noise event is the relaxation of a local delta-shaped disturbance of the stationary current carrier concentration by (one-dimensional) diffusion. For microscopical disturbances, the solution R of the diffusion equation can be assumed to be an averaged response to the disturbance:

$$R(u) = \bar{F}(u) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n F_i(u). \quad (5)$$

If the disturbances themselves are independent events, the relation

$$|\psi_{\bar{F}}(f)|^2 = \overline{|\psi_F(f)|^2}$$

is valid, because the interference terms in $\overline{|\psi_F(f)|^2}$ vanish. Hence in this case the spectral density may directly be calculated from the Fourier transform

$$\psi_{\bar{F}}(f) = \psi_R(f) = \int_{-\infty}^{+\infty} R(u) e^{-j\omega u} \, du \quad (3a)$$

through

$$S(f) = 2\lambda |\psi_R(f)|^2. \quad (4b)$$

b) One-dimensional Diffusion

In the one-dimensional case the flux Φ of the current carriers in an external field is given by

$$\Phi = -D \partial c / \partial x - E D c \quad (6)$$

with $E = \partial \bar{V} / \partial x$ (D = diffusion coefficient, $\bar{V} = V/kT$ = reduced potential energy, k = Boltzmann constant, T = absolute temperature). According to the definition of E , for $E > 0$ the direction of the external field E is opposite to the direction of the x -coordinate (see Figure 1). Throughout this paper we restrict ourselves to fields E which are constant

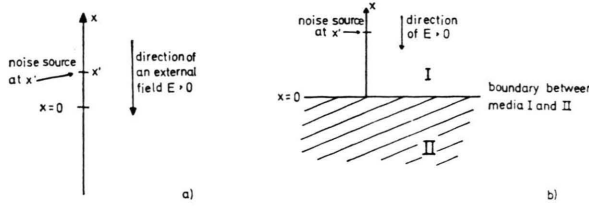


Fig. 1. Geometry of calculated diffusion models: a) infinite medium, b) semi-infinite medium.

in space and time. Then with the balance equation

$$\partial \Phi / \partial x = - \partial c / \partial t \quad (7)$$

and D independent of x we get the equation of diffusion

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} + E D \frac{\partial c}{\partial x}. \quad (8)$$

The first term on the right hand sides of (6) and (8) describes ordinary (Brownian) diffusion. The second term takes into account the effect of the external field.

Setting

$$c = \exp \left\{ -\frac{1}{4} E^2 D t \right\} \cdot \exp \left\{ -\frac{1}{2} E x \right\} \cdot v \quad (9)$$

one gets for v :

$$\frac{\partial v}{\partial t} = D \frac{\partial^2 v}{\partial x^2}. \quad (10)$$

Hence solutions c of (8) may be reduced upon solutions (v) of the usual diffusion equation with appropriate initial and boundary conditions for v .

3. Infinite Medium

In a first model calculation we consider an infinite one-dimensional diffusion regime. It is assumed that by a suitable experimental arrangement at $x=0$ fluctuations of concentration c or flux can be measured. Furthermore a 'noise source' is assumed to be located at $x=x'$ which creates statistically independent concentration perturbations around the stationary concentration c_s . In the following the index s is used to indicate stationary (time-independent) solutions. Quantities (c, Φ) without index represent deviations from the stationary state.

a) Solution of the Diffusion Equation

Let a single noise event begin at time $t=0$, i. e.

$$c(x-x', t=0) = \delta(x-x'). \quad (11 a)$$

With (9):

$$\exp \left\{ -\frac{1}{2} E x \right\} \cdot v(x-x', t=0) = \delta(x-x'). \quad (11 b)$$

The solution v of (10) with (11) in the infinite region is (see e. g. ²³):

$$v = \frac{1}{2 \sqrt{\pi D t}} \exp \left\{ -\frac{(x-x')^2}{4 D t} \right\} \cdot \exp \left\{ \frac{1}{2} E x' \right\}. \quad (12 a)$$

The factor $\exp \left\{ \frac{1}{2} E x' \right\}$ in (12 a) occurs because the basic initial condition (11 a) holds for concentration c . We get with (9)

$$c = \exp \left\{ -\frac{E}{2} (x-x') \right\} \cdot \exp \left\{ -\frac{E^2}{4} D t \right\} \cdot \frac{1}{2 \sqrt{\pi D t}} \exp \left\{ -\frac{(x-x')^2}{4 D t} \right\}. \quad (12 b)$$

The resulting flux may be derived from (12 b) with (6):

$$\Phi = \left(\frac{x-x'}{2 D t} - \frac{E}{2} \right) c. \quad (13)$$

b) Spectral Density of Fluctuations in Concentrations

According to Carson's theorem the spectral density $S_c(\omega)$ of fluctuations of concentration at $x=0$ which are generated by the 'noise source' at x' is given by the Fourier transform $\psi_c(\omega)$ of c through

$$S_c(\omega) \propto |\psi_c(\omega)|^2. \quad (5 a)$$

The Fourier transform $\psi_c(\omega)$ is according to (12 b) for $x=0$:

$$\psi_c(\omega) = \exp \left\{ +\frac{E}{2} x' \right\} \int_0^\infty \exp \left\{ -\frac{E^2}{4} D t \right\} \cdot \exp \left\{ -\frac{x'^2}{4 D t} \right\} \frac{1}{2 \sqrt{\pi D t}} \cdot e^{-j\omega t} dt. \quad (14)$$

The integral in (14) may be evaluated in closed form (see e. g. in ²⁴ integral number 337.6 with the substitution $t=x^2$). The result is

$$\begin{aligned} \text{Re } \psi_c(\omega) &= \frac{1}{E D} \exp \left\{ \frac{1}{2} E x' \right\} \\ &\cdot \exp \left\{ -\frac{1}{2} E \left| x' \right| \left[\frac{1}{2} (\sqrt{1+\alpha^2} + 1) \right]^{1/2} \right\} \\ &\cdot (1+\alpha)^{-1/4} \left[\frac{\cos}{\sin} (\varphi + 2 \varrho \sigma \sin \varphi) \right] \end{aligned} \quad (15)$$

with

$$\begin{aligned} \alpha &= \frac{4 \omega}{E^2 D}, \quad \varphi = \frac{1}{2} \arctg \alpha, \\ \varrho &= \sqrt[4]{1+\alpha^2}, \quad \sigma = \frac{|x'|}{2 \sqrt{D}}. \end{aligned}$$

Hence for the spectral density holds

$$S_c(\omega) \propto E^2 D^2 (\operatorname{Re} \psi_c^2 + \operatorname{Im} \psi_c^2) \quad (16)$$

$$= \frac{1}{\sqrt{1+\alpha^2}} \cdot \exp \left\{ E x' (1 \mp [\tfrac{1}{2}(\sqrt{1+\alpha^2} + 1)]^{1/2}) \right\}$$

with \mp for $x \geq 0$.

c) Spectral Density of Fluctuations in Flux

The corresponding spectral density $S_\Phi(\omega)$ of the fluctuations in flux Φ can analogously be calculated from the time course (13) of a single noise event. The integrals occurring in the Fourier transform $\psi_\Phi(\omega)$ can also be evaluated in closed form (integrals number 337.6 and 337.8 in ²⁴). The result is

$$\begin{aligned} \operatorname{Re} \psi_\Phi(\omega) &= \frac{1}{2D} \exp \left\{ \frac{E}{2} (x - x') \right\} \\ &\cdot \exp \left\{ -\frac{E}{2} |x - x'| [\tfrac{1}{2}(\sqrt{1+\alpha^2} + 1)]^{1/2} \right\} \\ &\left[- (1 + \alpha)^{-1/4} \frac{\cos}{\sin} (\varphi + 2 \varrho \sigma \sin \varphi) \right. \\ &\quad \left. + \operatorname{sign}(x - x') \cdot \frac{\cos}{\sin} (2 \varrho \sigma \sin \varphi) \right] \end{aligned} \quad (18)$$

with $\alpha, \varrho, \sigma, \varphi$ as in (15).

Carson's theorem for the spectral density S_Φ of flux fluctuations at $x = 0$ gives:

$$S_\Phi(\omega) \propto \exp \left\{ E x' (1 \mp [\tfrac{1}{2}(\sqrt{1+\alpha^2} + 1)]^{1/2}) \right\} \quad (19)$$

$$\cdot \left[\frac{1}{\sqrt{1+\alpha^2}} \pm \frac{2[\tfrac{1}{2}(\sqrt{1+\alpha^2} + 1)]^{1/2}}{\sqrt{1+\alpha^2}} + 1 \right].$$

d) Discussion

Figure 2 represents S_c and S according to (16), (19) as functions of the dimensionless quantity α

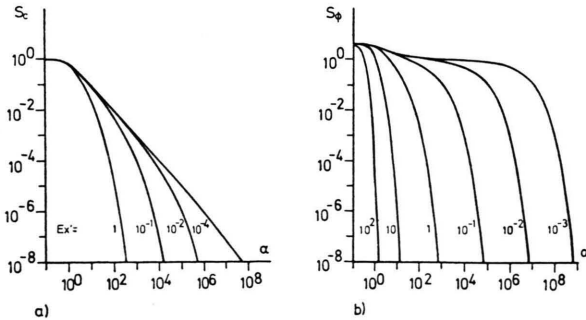


Fig. 2. Results for diffusion in an infinite medium with a delta-shaped noise source at x' . a) Spectral density $S_c(\alpha)$ of resulting fluctuations in concentration c at $x=0$ (arbitrary units). b) Spectral density $S_\Phi(\alpha)$ of fluctuations in flux Φ at $x=0$.

which is linearly coupled to frequency ω , for different values of the dimensionless parameter $E x'$. In both cases the spectral densities correspond to white noise for $\omega(\alpha) \rightarrow 0$. For $\omega(\alpha) \rightarrow \infty$, the spectral densities tend zero, but in contrast to the spectral density S_Φ of the flux S_c exhibits 1/f behaviour in an intermediate region. For sufficiently small values of $E x'$ this 1/f behaviour extends over many orders of magnitude. The reason why the fluctuations of flux do not exhibit 1/f behaviour is the different time-dependence of Φ in (13). According to (6) the flux Φ is determined not only by the concentration c but also by the gradient of c .

The results (16) and (19) may also be used in cases where the 'noise source' is extended over some region enclosing $x=0$. Under the assumption that between \bar{x} and $-\bar{x}$ fluctuations (11 a) in concentration c occur independently with the same probability for all $x' \in [-\bar{x}, \bar{x}]$, the spectral densities \bar{S}_c , \bar{S}_Φ of the resulting noise are given by simple integration over x' from $-\bar{x}$ to \bar{x} . The results are

$$\bar{S}_c \propto \frac{1}{\sqrt{1+\alpha^2}} \left[\left(\frac{1}{\beta-1} \right) \cdot (1 - \exp \{ -E \bar{x}(\beta-1) \}) \right. \\ \left. + \left(\frac{1}{\beta+1} \right) (1 - \exp \{ -E \bar{x}(1+\beta) \}) \right] \quad (16 a)$$

$$\text{with} \quad \beta = [\tfrac{1}{2}(\sqrt{1+\alpha^2} + 1)]^{1/2}$$

and

$$\begin{aligned} \bar{S}_\Phi \propto & \left(\frac{1}{\sqrt{1+\alpha^2}} + \frac{2\beta}{\sqrt{1+\alpha^2}} + 1 \right) \cdot \left(\frac{1}{\beta-1} \right) \\ & \cdot (1 - \exp \{ -E \bar{x}(\beta-1) \}) \\ & + \left(\frac{1}{\sqrt{1+\alpha^2}} - \frac{2\beta}{\sqrt{1+\alpha^2}} + 1 \right) \cdot \left(\frac{1}{1+\beta} \right) \\ & \cdot (1 - \exp \{ -E \bar{x}(\beta+1) \}) \end{aligned} \quad (19 a)$$

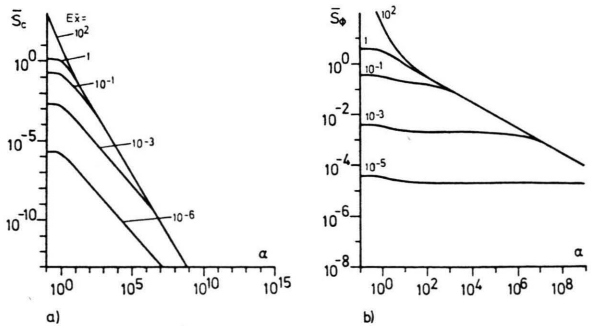


Fig. 3. Results for diffusion in an infinite medium, with a noise source extended from $-\bar{x}$ to $+\bar{x}$. a) Spectral density $\bar{S}_c(\alpha)$ of fluctuations in concentration c at $x=0$. b) Spectral density $\bar{S}_\Phi(\alpha)$ of fluctuations in flux Φ at $x=0$.

\bar{S}_c and \bar{S}_Φ from (16 a) and (19 a) are represented in Figures 3 a, b. Again, for sufficiently small $E \bar{x}$, the spectral density \bar{S}_c shows a 1/f behaviour but \bar{S}_Φ does not. For $\omega(\alpha) \rightarrow \infty$, S_c decays as $1/\omega^{3/2}$ in agreement with the 'universal 3/2 power law' for fluctuations arising from diffusional mechanism stated e. g. by Lax and Mengert¹⁸.

4. Semi-infinite Medium

a) Boundary Condition

We now consider a system where passage of current carriers from a medium I into a second medium II is controlled by a boundary condition at the contact $x=0$ (see Fig. 1 b) between the two media. The one-dimensional flux in medium I ($x>0$) is determined by the diffusion coefficient D and field E according to Equation (6). The concentration c_{II} of carriers in medium II near the boundary is assumed to be constant in time. For the passage of carriers through the boundary at $x=0$ we postulate the following boundary conditions for the total flux $\Phi_{\text{total}} = \Phi_s + \Phi$:

$$\Phi_{\text{total}}(x=0) = -(A_I c_I - A_{II} c_{II}) - E(B_I c_I) \quad (20)$$

where A_I , A_{II} , B_I are (positive) constants, c_I is the total concentration in medium I at $x = +0$.

The sign of Φ is negative if the flux is directed from medium I to medium II. If $A_I \neq A_{II}$, the equilibrium concentrations c_I and c_{II} for vanishing flux and field are not equal (partition coefficient $\neq 1$). Then the stationary solution Φ_s , c_s must satisfy

$$\begin{aligned} \Phi_s(x=0) = & -[A_I c_I(x=+0) - A_{II} c_{II}] \\ & - E[B_I c_s(x=+0)] \end{aligned} \quad (20 a)$$

and for the deviation Φ , c from stationarity holds:

$$\Phi(x=+0) = -A \cdot c(x=+0), \quad A = A_I + B_I E. \quad (20 b)$$

(20) means that the number of current carriers passing from I into II increases with the number of the carriers at the boundary. This boundary condition should be suitable to describe many systems in which a contact exists between two different media. In the theory of heat transfer the condition (20 b) is known as 'radiation condition'. A number of examples where this condition has been used, may be found in the book of Carslaw and Jaeger²³.

For the auxiliary function v (20) yields with (7), (9)

$$\frac{\partial v}{\partial x(x=+0)} = h v, \quad h = \frac{A}{D} - \frac{E}{2}. \quad (20 c)$$

The results for diffusion in an infinite medium have given a basis for the model calculations in this chapter: In the infinite medium the concentration c shows 1/f behaviour, while the flux Φ does not. On the other hand, in systems with contacts between different media, flux through this contact and concentrations are connected by the boundary condition (20). So we may expect that under the condition (20) the flux exhibits also a 1/f behaviour.

b) Spectral Density

The fluctuating quantity of interest is the flux $\Phi(x=0)$ through the boundary. Again we first assume the noise to be generated by a 'noise source' at x' ($x'>0$). The initial condition for a single noise event starting at $t=0$ is [c. f. (11)]:

$$c(x, t=0) = \exp\left\{-\frac{1}{2} E x'\right\} v(x, t=0) = \delta(x-x'). \quad (21)$$

The solution for a quantity v , obeying a usual diffusion equation (10) in a semi-infinite medium with a radiation boundary condition (20 c) and the initial condition (21) has been found by Bryan²⁵ (see also Carslaw²⁶ or Carslaw and Jaeger²³, page 358–359):

$$\begin{aligned} v = \exp\left\{\frac{E}{2} x'\right\} x' \left[\frac{1}{2 \sqrt{\pi D t}} \cdot \left(\exp\left\{-\frac{(x-x')^2}{4 D t}\right\} \right. \right. \\ \left. \left. + \exp\left\{-\frac{(x+x')^2}{4 D t}\right\} \right) \right. \\ \left. - \frac{h}{\sqrt{\pi D t}} \int_0^\infty \exp\{-h \xi\} \cdot \exp\left\{-\frac{(x+x'+\xi)^2}{4 D t}\right\} d\xi \right]. \end{aligned} \quad (22)$$

This yields for the concentration c at the boundary $x=+0$:

$$\begin{aligned} c(x=+0) = \exp\left\{\frac{E}{2} x'\right\} \cdot \exp\left\{-\frac{E^2}{4} D t\right\} \cdot \frac{1}{\sqrt{\pi D t}} \\ \cdot \left[\exp\left\{-\frac{x'^2}{4 D t}\right\} - h \int_0^\infty \exp\{h \xi\} \exp\left\{-\frac{(x'+\xi)^2}{4 D t}\right\} d\xi \right]. \end{aligned} \quad (23)$$

According to condition (20) the flux Φ at $x=+0$ is given by the concentration c at $x=+0$. Therefore the spectral density $S(\omega)$ of concentration as well as of the current fluctuations may be determined by the Fourier transform $\psi_c(\omega)$ of the time course

of concentration c generated by a 'noise event' (21) at x' . The first term on the right-hand side of (23) yields the same type of integrals in the Fourier transform as in (14). The second term leads to

$$\int_0^\infty dt \exp\{-j\omega t\} \left[\exp\left\{\frac{E}{2}x'\right\} \exp\left\{-\frac{E^2}{4}Dt\right\} \frac{1}{\sqrt{\pi Dt}} \cdot h \int_0^\infty \exp\{-h\xi\} \exp\left\{-\frac{(x'+\xi)^2}{4Dt}\right\} d\xi \right].$$

It may be evaluated as follows: First we perform the time integration analogously as in (14). Then the integral over ξ may be calculated in closed form (see e. g. ²⁴ integral number 335, 1 a and 2 a). The final result for ψ_c is

$$\begin{aligned} \text{Re } \psi_c &= \frac{2}{ED} \cdot (1+\alpha^2)^{-1/4} \\ -\text{Im } \psi_c &= \frac{2}{ED} \cdot (1+\alpha^2)^{-1/4} \cdot \exp\left\{\frac{E}{2}x' \left[1 - \frac{1}{2}(\sqrt{1+\alpha^2} + 1)\right]^{1/2}\right\} \\ &\cdot \left\{ \cos \kappa - h \left(\frac{B \cos \kappa + D \sin \kappa}{B^2 + D^2} \right) \right\} \end{aligned} \quad (24)$$

with the abbreviations $\alpha, \varrho, \sigma, \varphi$ as in (15) and

$$\begin{aligned} \kappa &= \varphi + 2\varrho\sigma \sin \varphi, \\ B &= h + \frac{E}{2} \left[\frac{1}{2}(\sqrt{1+\alpha^2} + 1) \right]^{1/2}, \\ D &= \frac{E}{2} \left[\frac{1}{2}(\sqrt{1+\alpha^2} - 1) \right]^{1/2}. \end{aligned} \quad (24a)$$

Then with Carson's theorem

$$\begin{aligned} S_c \propto S_\phi = S &\propto \frac{D^2 E^2}{4} (\text{Re } \psi_c^2 + \text{Im } \psi_c^2) \\ &= \frac{1}{\sqrt{1+\alpha^2}} \exp\left\{-E x' \left[\frac{1}{2}(\sqrt{1+\alpha^2} + 1) \right]^{1/2} - 1\right\} \\ &\cdot \left[\frac{\frac{E^2}{4 h^2} \sqrt{1+\alpha^2}}{\frac{E^2}{4 h^2} \sqrt{1+\alpha^2} + 1 + \frac{E}{h} \left[\frac{1}{2}(\sqrt{1+\alpha^2} + 1) \right]^{1/2}} \right]. \end{aligned} \quad (25)$$

Analogously as above we can regard cases where the 'noise source' is extended over a region from $x=0$ to $\bar{x}>0$. By integration of S over x' from 0 to \bar{x} in (25) the spectral density \bar{S} of the resulting noise is:

$$\begin{aligned} \bar{S} &\propto \frac{1}{\sqrt{1+\alpha^2}} \left(\frac{1}{\beta-1} \right) [1 - \exp\{-E\bar{x}(\beta-1)\}] \\ &\cdot \left[\frac{\frac{E^2}{4 h^2} \sqrt{1+\alpha^2}}{1 + \frac{E}{h} \beta + \frac{E^2}{4 h^2} \sqrt{1+\alpha^2}} \right]. \end{aligned} \quad (25a)$$

S and \bar{S} are drawn in Fig. 4 a, b for the value $E/h = 5$. According to (20), (20 a, b, c) for $(A_I, B_I, E > 0)$ h is greater than $-E/2$ and hence $E/h > -2$. For all possible values of E/h the spectral densities S, \bar{S} show a similar behaviour. Especially for sufficiently small $E x', E \bar{x}$ the spectral densities S, \bar{S} exhibit a $1/f$ behaviour over many orders of magnitude in $\omega(\alpha)$. Similarly as \bar{S}_c in (16 a) \bar{S} behaves asymptotically $\propto 1/\omega^{3/2}$ for $\omega \rightarrow \infty$.

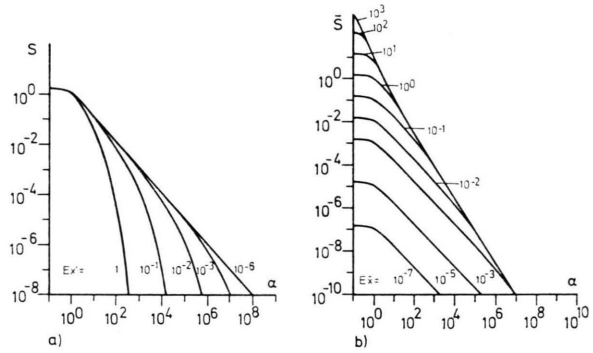


Fig. 4. Results for diffusion in a semi-infinite medium. Spectral density $S(\alpha)$ of fluctuations in concentration and flux at the boundary $x=0$ (arbitrary units). a) Delta-shaped noise source at x' ; b) noise source extended from 0 to \bar{x} .

5. Discussion

In electrical systems a relevant fluctuating quantity is current and not concentration. Our model calculations have shown one especially interesting result: An essential condition for the occurrence of a $1/f$ behaviour of current may be the existence of a contact, i. e. a flux controlling boundary between two media. The special structure of the boundary condition itself has no decisive influence on the $1/f$ behaviour, at least within the wide frame of the radiation boundary condition (20). This may be seen from the fact that in (25) and (25 a) the special value of h , and therefore of A in (20 b) does not influence essentially the shape of $S(\omega), \bar{S}(\omega)$.

As mentioned in the introduction, the stimulation for performing our model calculations came from the recent experiments by Dorset and Fishman¹⁴, who have measured $1/f$ fluctuations in porous structures. For pore lengths of the order $2 \cdot 10^{-3}$ cm [Polyethylene terephthalate ("Mylar") films] an applied voltage of 100 mV and a diffusion coefficient of 10^{-5} cm²/sec, one can roughly estimate the frequency ω as a function of the dimensionless

parameter α : According to (15)

$$\omega = \alpha \cdot \frac{1}{4} E^2 D \quad (26)$$

and hence with the definition of E [see (6)] one gets $\omega[\text{Hz}] \approx \alpha$. As to be seen from Fig. 4 the theoretical deviation from the 1/f behaviour, i. e. the transition to white noise, generally occurs for $\alpha \approx 1$. Dorset and Fishman have measured 1/f noise down to frequencies of ≈ 1 Hz. Hence their results for mylar films are still compatible with the estimated 1/f frequency region, if we assume the decisive quantity for current fluctuations to be the transition from the pores into the aqueous solution, where the sources of these fluctuations are spatial fluctuations of ion concentration within the pores.

The situation is more unfavourable for pore lengths $\approx 10^{-5}$ cm as used by Dorset and Fishman in mica (Muscovite) sheets, collodion or polyvinyl formal films as well as for the observed 1/f fluctuations in biological membranes (pore length $\approx 10^{-7}$ cm). In these cases the 1/f frequency regions are not compatible. The white noise limit begins at higher frequencies than 1 Hz.

Possibly our model may be applied also for these systems, if we assume that the interior of the pores is medium II (see Fig. 1 b) and the aqueous phase outside is medium I. The greatest part of the applied voltage falls off within the pores. But it can be assumed, that a small electric field is also in the immediate vicinity before the pore mouths.

Hence our model calculations indicate, that concentration perturbations before the pore mouth may

lead to 1/f noise in the needed frequency region: The electric field before the pore mouths is much smaller than within the pores. So according to relation (26) a 1/f-behaviour is expected down to very low frequencies. The diffusion regime is now three-dimensional. But the field lines are directed into the pore mouths, and decisive for the current fluctuations is the effect of local concentration perturbations on the transition of the current carriers through the pore mouths. Hence one expects that the dimensionality of diffusion is reduced and the one-dimensional model calculations may be regarded as a qualitatively good approximation of reality. A detailed investigation of this case will be done in a later paper.

Possibly our considerations may be applied also for other 1/f noise sources, e.g. semi conductor diodes, transistors or carbon microphones, where always the flux of the charge carriers (electrons) might be controlled by contacts between different media.

In this paper we have restricted attention to the phenomenon of 1/f behaviour itself. We have omitted the problem of the *intensity*, which will be subject to future work.

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